

Hiemenz Flow through a Porous Medium of a Non-Newtonian Rivlin-Ericksen Fluid with Heat Transfer

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Abstract

The steady laminar flow through a porous medium of an incompressible non-Newtonian Rivlin-Ericksen fluid impinging normal to a plane wall with heat transfer is investigated. A numerical solution for the governing nonlinear momentum and energy equations is obtained. The effect of the porosity of the medium and the characteristics of the non-Newtonian fluid on both the flow and heat transfer is outlined.

Key Words: Hiemenz Flow, Porous Medium, Non-Newtonian Fluid, Heat Transfer

1. Introduction

The two-dimensional flow of a fluid near a stagnation point is a classical problem in fluid mechanics. It was first examined by Hiemenz [1] who demonstrated that the Navier-Stokes equations governing the flow can be reduced to an ordinary differential equation of third order using similarity transformation. Owing to the nonlinearities in the reduced differential equation, no analytical solution is available and the nonlinear equation is usually solved numerically subject to two-point boundary conditions, one of which is prescribed at infinity.

Later the problem of stagnation point flow was extended in numerous ways to include various physical effects. The effect of suction on Hiemenz problem has been considered in the literature. Schlichting and Bussman [2] gave the numerical results first. More detailed solutions were later presented by Preston [3]. An approximate solution to the problem of uniform suction is given by Ariel [4].

The study of heat transfer in boundary layer flows is

of importance in many engineering applications such as the design of thrust bearings and radial diffusers, transpiration cooling, drag reduction, thermal recovery of oil, etc. [5]. Massoudi and Ramezan [5] used a perturbation technique to solve for the stagnation point flow and heat transfer of a non-Newtonian fluid of second grade. Their analysis is valid only for small values of the parameter that determines the behavior of the non-Newtonian fluid. Later Massoudi and Ramezan [6] extended the problem to nonisothermal surface. Garg [7] improved the solution obtained by Massoudi [6] by computing numerically the flow characteristics for any value of the non-Newtonian parameter using a pseudo-similarity solution.

Non-Newtonian fluids were considered by many researchers. Thus, among the non-Newtonian fluids, the solution of the stagnation point flow, for viscoelastic fluids, has been given by Rajeshwari and Rathna [8], Beard and Walters [9], Teipel [10], Ariel [11], and others; for power-law fluid by Djukic [12]; and for micropolar fluids by Nath [13]. Stagnation point flow of a non-Newtonian second grade fluid was studied by Teipel [14] and Ariel [15]. In hydromagnetics, Attia [16] introduced the influence of an external uniform magnetic field on

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the flow of a second grade fluid.

The purpose of the present paper is to study the steady laminar flow through a porous medium of an incompressible non-Newtonian second grade fluid at a two-dimensional stagnation point with heat transfer. The wall and stream temperatures are assumed to be constants. The flow in the porous media deals with the analysis in which the differential equation governing the fluid motion is based on the Darcy's law which accounts for the drag exerted by the porous medium [17–19]. A numerical solution is obtained for the governing momentum and energy equations using finite difference approximations which takes into account the asymptotic boundary conditions at infinity. The numerical solution are used to determine the flow and heat characteristics for the whole range of the non-Newtonian fluid characteristics, the porosity parameter and the Prandtl number.

2. Formulation of the Problem

Consider the two-dimensional stagnation point flow of an incompressible non-Newtonian Rivlin-Ericksen fluid impinging perpendicular to a plane directed along the x -axis. This is an example of a plane potential flow that arrives from the y -axis and impinges on a flat wall placed at $y = 0$, divides into two streams on the wall and leaves in both directions. The viscous flow must adhere to the wall, whereas the potential flow slides along it. (u, v) are the components of velocity at any point (x, y) for the viscous flow whereas (U, V) are the velocity components for the potential flow. The velocity distribution in the frictionless flow in the neighborhood of the stagnation point is given by

$$U(x) = ax, V(y) = -ay$$

where the constant $a(> 0)$ is proportional to the free stream velocity far away from the stretching surface. A second grade fluid is defined such that the Cauchy stress tensor is related to the fluid motion in the following manner [16]:

$$T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 \quad (1)$$

where p denotes the pressure, I is the identity tensor, μ is the shear viscosity of the fluid, α_1 and α_2 are scalar constants named as normal stress moduli, and A_1 and A_2 are the first two-Rivlin-Ericksen tensors given by,

$$\vec{A}_1 = \text{grad} \vec{v} + (\text{grad} \vec{v})^T \quad (2)$$

$$\vec{A}_2 = \frac{d}{dt} A_1 + A_1 (\text{grad} \vec{v}) + (\text{grad} \vec{v})^T A_1 \quad (3)$$

where $\vec{v}$ is the velocity vector and d/dt is the material time derivative, which is defined as follows:

$$\frac{d}{dt}(\cdot) = \frac{\partial(\cdot)}{\partial t} + [\text{grad}(\cdot)]\vec{v} \quad (4)$$

For $\alpha_1 = \alpha_2 = 0$, Eq. (1) describes a common Newtonian fluid. Then, A_1 represents the usual deformation tensor. All the stress components have to be introduced into the equations of motion. Here, we consider the case $\alpha_2 = 0$, i.e. the case of a reduced Rivlin-Ericksen fluid [16]. Then, for the two-dimensional steady-state flows, the continuity and momentum equations, using the usual boundary layer approximations and by introducing the stress components and porosity force, reduce to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (5)$$

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = U \frac{dU}{dx} + \mu \left(\frac{\partial^2 u}{\partial y^2} \right) - \frac{\mu}{K1} u + \alpha_1 \left(\frac{\partial u}{\partial y} \frac{\partial^2 u}{\partial y^2} + v \frac{\partial^3 u}{\partial y^3} + \frac{\partial}{\partial x} \left(u \frac{\partial^2 u}{\partial y^2} \right) \right) = 0 \quad (6)$$

where ρ is the density of the fluid, $K1$ is the Darcy permeability [17–19], and $U(x)$ is the potential flow velocity over the body surface.

Using the boundary layer approximations and neglecting the viscous dissipation, the equation of energy for temperature T is given by [5,6],

$$\rho c_p \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \frac{\partial^2 T}{\partial y^2} \quad (7)$$

where T is the temperature of the fluid, c_p is the specific heat capacity at constant pressure of the fluid, and k is the thermal conductivity of the fluid. A similarity solution exists if the wall and free-stream temperatures, T_w and T_∞ are constants – a realistic approximation in typical stagnation point heat transfer problems [5,6].

The boundary conditions are

$$y = 0 : u = 0, v = 0 \quad (8a)$$

$$y \rightarrow \infty : u \rightarrow ax, \frac{\partial u}{\partial y} \rightarrow 0 \quad (8b)$$

$$y = 0 : T = T_w \quad (9a)$$

$$y \rightarrow \infty : T \rightarrow T_\infty \quad (9b)$$

A little inspection shows that boundary-layer equations (5)–(6) admit a similarity solution

$$u(x, y) = axf'(\eta), v(x, y) = -\sqrt{av}f(\eta), \eta = \sqrt{a/v}y \quad (10)$$

where the prime denotes differentiation with respect to η and $v = \mu/\rho$. By introducing the non-dimensional variable

$$\theta = \frac{T - T_\infty}{T_w - T_\infty}$$

and using Eq. (10), we find that Eq. (5) is identically satisfied and Eqs. (6)–(7) reduce to,

$$K(ff^{iv} - 2f'f''' + f''^2) - f''' - fff'' + f'^2 - M(1 - f') - 1 = 0 \quad (11)$$

$$\theta'' + \text{Pr} f \theta' = 0 \quad (12)$$

$$f(0) = 0, f'(0) = 0, f'(\infty) = 1, f''(\infty) = 0 \quad (13)$$

$$\theta(0) = 1, \theta(\infty) = 0 \quad (14)$$

where Pr is the Prandtl number, $\text{Pr} = \mu c_p/k$, K is the di-

mensionless normal stress modulus, $K = \alpha_1 a/\mu$, $M = v/(aK1)$ is the porosity parameter, and the prime denotes differentiation with respect to η .

The heat transfer at the wall is computed from Fourier's law [5,6] as follows;

$$q_w = -k \left(\frac{\partial T}{\partial y} \right)_{y=0} = -k(T_\infty - T_w) \sqrt{a/v} G(\text{Pr})$$

where G is the dimensionless heat transfer rate which is given by

$$G^{-1} = \int_0^\infty d\eta \exp(-2\text{Pr} \int_0^\eta f ds)$$

The equations to be solved are Eqs. (11)–(14). The flow Eqs. (11) and (13) are decoupled from the energy Eqs. (12) and (14), and need to be solved before the latter can be solved. The flow Eq. (11) constitutes a non-linear, non-homogeneous boundary value problem (BVP). In the absence of an analytical solution of a problem, a numerical solution is required. The flow Eqs. (11) and (13) are solved numerically using finite difference approximations. A quasi-linearization technique is first applied to replace the non-linear terms at a linear stage, with the corrections incorporated in subsequent iterative steps until convergence. The quasi-linearized form of Eq. (11) is,

$$\begin{aligned} K(f_n f_{n+1}^{iv} + f_n^{iv} f_{n+1} - f_n f_n^{iv} - 2(f_n' f_{n+1}''' + f_n''' f_{n+1}' - f_n'' f_n') \\ + 2f_n'' f_{n+1}'' - f_n''^2) - f_{n+1}''' - f_n f_{n+1}'' - f_n'' f_{n+1} \\ + f_n'' f_n + 2f_n' f_{n+1}' - f_n'^2 - M(1 - f_{n+1}') - 1 = 0 \end{aligned}$$

where the subscript n or $n + 1$ represents the n th or $(n + 1)$ th approximation to the solution. Then, the different terms are replaced by their second order central difference approximations. An iterative scheme is used to solve the quasi-linearized system of difference equations. The solution for the Newtonian case is chosen as an initial guess, where the variables with the n th subscript are given corresponding values, and the iterations are continued till convergence within prescribed accuracy. Finally, the resulting block tri-diagonal system was solved using generalized Thomas' algorithm.

The energy Eq. (12) is a linear second order ordinary differential equation with variable coefficient, $f(\eta)$, which is known from the solution of the flow Eqs. (11) and (13) and the Prandtl number Pr is assumed constant. Equation (12) is solved numerically under the boundary condition (14) using central differences for the derivatives and Thomas' algorithm for the solution of the set of discretized equations. The resulting system of equations has to be solved in the infinite domain $0 < \eta < \infty$. A finite domain in the η -direction can be used instead with η chosen large enough to ensure that the solutions are not affected by imposing the asymptotic conditions, given in Eqs. (13) and (14), at a finite distance. Grid-independent studies show that the computational domain $0 < \eta < \eta_\infty$ can be divided into intervals each of uniform step size which equals 0.02. This reduces the number of points between $0 < \eta < \eta_\infty$ without sacrificing accuracy. The value $\eta_\infty = 10$ was found to be adequate for all the ranges of parameters studied here. Results for the flow and temperature distributions are obtained for various values of the parameters K , M , and Pr .

3. Results and Discussion

Figure 1 presents the velocity profiles $f'(\eta)$ for various values of K and for $M = 0$ and 1. The figure shows that the velocity boundary layer thickness increases with K . On the other hand, the velocity boundary layer thickness decreases with increasing M due to its damping effect, while the effect of M becomes more pronounced with higher values of K . Table 1 shows the effect of M and K on the wall shear stress $f''(0)$. Increasing the pa-

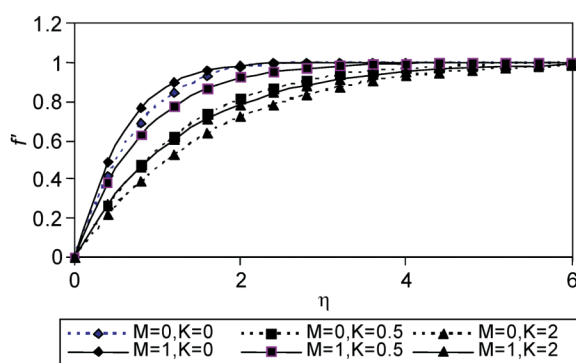


Figure 1. Profiles of f' for various values of M and K .

rameter K decreases the wall shear stress. On the other hand, the wall shear stress increases steadily with the increment in M for all values of K . This can be attributed to the fact that increasing M increases the tangential velocity and its wall slope. This causes the skin-friction coefficient to increase as M increases. The results presented in Table 1 for the case $M = 0$, show good agreement with the previous results presented in refs. [5–7].

Figure 2 presents the temperature profiles $\theta(\eta)$ for various values of K and for $M = 0$ and 1 and $Pr = 0.7$. It is clear from the figure that the thermal boundary layer thickness increases with K . Also, the absolute value of the temperature gradient at the wall increases with K . The figure indicates also the influence of the porosity on the temperature distribution and shows that the thermal boundary layer thickness decreases with increasing M as a result of preventing fluid at near-ambient temperature from reaching the surface.

Figure 3 shows the temperature profiles for various values of Pr and for $M = 0.5$ and $K = 1.0$. The figure brings out clearly the effect of the Prandtl number on the thermal boundary layer thickness. Increasing Pr decreases the thermal boundary layer thickness. In general,

Table 1. Variation of the wall shear stress $f''(0)$ with K and M ($Pr = 0.5$)

M	$K = 0$	$K = 0.5$	$K = 1$	$K = 5$	$K = 10$
0	1.2326	0.9025	0.7528	0.4130	0.3031
0.5	1.3283	0.9810	0.8215	0.4555	0.3355
1	1.5840	1.1860	1.0000	0.5641	0.4180
1.5	1.9374	1.4666	1.2420	0.7092	0.5277
2	2.3449	1.7854	1.5175	0.8727	0.6509

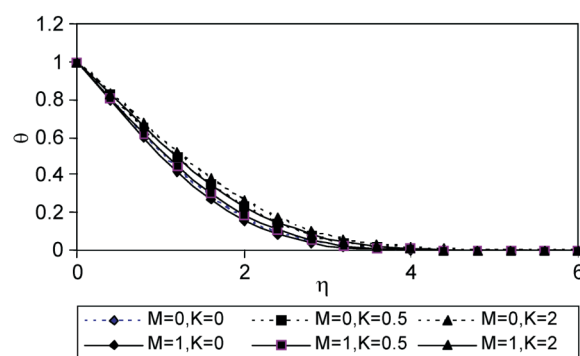


Figure 2. Profiles of θ for various values of M and K .

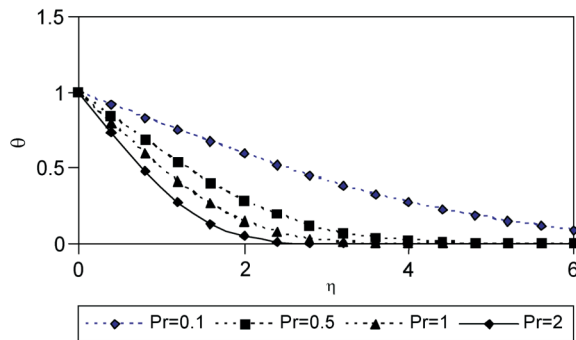


Figure 3. Profiles of θ for various values of Pr .

the velocity boundary layer is thicker than the thermal boundary layer when $Pr > 1$, because viscous diffusion exceeds conduction effects and the opposite is also true. However, for $Pr < 1$, the thermal boundary layer is thicker than the velocity boundary layer.

Table 2 presents the variation of $G(Pr)$ with Pr for various values of K and for $M = 0.1$. It is clear that increasing Pr increases G . However, increasing K decreases G and the influence of K becomes more pronounced for higher values of Pr . Table 3 presents the variation of $G(Pr)$ with Pr for various values of M and for $K = 1.0$. It indicates that G increases with the increment in M .

Table 2. Variation of the wall heat transfer $G(Pr)$ with K and Pr ($M = 0.1$)

K	$Pr = 0.05$	$Pr = 0.1$	$Pr = 0.5$	$Pr = 1$	$Pr = 2$
0	0.1667	0.2206	0.4354	0.5739	0.7496
0.5	0.1607	0.2098	0.4041	0.5285	0.6861
1	0.1569	0.2032	0.3864	0.5033	0.6513
5	0.1440	0.1806	0.3304	0.4258	0.5464
10	0.1377	0.1695	0.3039	0.3899	0.4986

Table 3. Variation of the wall heat transfer $G(Pr)$ with M and Pr ($K = 1$)

M	$Pr = 0.05$	$Pr = 0.1$	$Pr = 0.5$	$Pr = 1$	$Pr = 2$
0	0.1569	0.2031	0.3861	0.5029	0.6507
0.5	0.1580	0.2050	0.3918	0.5114	0.6631
1	0.1605	0.2095	0.4051	0.5313	0.6918
1.5	0.1632	0.2145	0.4202	0.5539	0.7249
2	0.1657	0.2190	0.4342	0.5753	0.7563

4. Conclusion

The two-dimensional stagnation point flow through a porous medium of a viscous incompressible non-Newtonian Rivlin-Ericksen fluid with heat transfer is studied. A numerical solution for the governing equations was obtained that allows the computation of the flow and heat transfer characteristics for various values of the non-Newtonian parameter K , the porosity parameter M , and the Prandtl number Pr . The results indicate that increasing the parameter K increases both the velocity and thermal boundary layer thickness while the porosity of the medium has a reversed effect. On the other hand, the porosity results in the increment of the wall shear stress. The results show that the heat transfer at the wall increases with M and Pr while it decreases with K .

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